The incremental work of total deformation is

$$dw_d = 4v\tau \, d\epsilon_x/3.$$

(4.16)

The incremental work of elastic deformation is

$$dw_{de} = 8v\tau \, d\gamma^e/3 = (4v\tau/3\mu) \, dt = (2v/3\mu) \, d(\tau^2).$$

(4.17)

The incremental work of plastic deformation is obtained by subtracting equation (4.17) from equation (4.16):

$$dw_p = (4/3)v\tau(d\epsilon_x - d\tau/\mu) = (8/3)v\tau \, d\gamma^p$$

(4.18)

where $d\gamma^p = (d\epsilon_x^p - d\epsilon_y^p)/2$.

If we assume that there is always an elastically-strained state imbedded in every deformed state, and that this elastic state is inherently reversible and recoverable, it follows that an internal energy function exists which depends on elastic strains, entropy, and perhaps internal variables. Neglecting internal variables, increments in the internal energy function can be written[19]

$$de = T \, dS + dw_e.$$

(4.19)

According to equation (4.19) and the foregoing assumptions, internal energy can be expressed in terms of entropy and the elastic strains. For example,

$$e = e(S, \epsilon_x, \epsilon_x^p).$$

(4.20)

The appearance of ϵ_x^p in this potential is illusory. It does not imply a physical dependence of e on plastic strain. Nonetheless, $(\partial e/\partial \epsilon_x^p)_{\epsilon_x S}$ exists and must be counted in applications.

Referring now to equation (4.1), we have $\xi = \epsilon_x^p$, $p_x = p^*$, $a^* = a$, $I^* = (\partial p_x/\partial S)_{\rho,\epsilon_p}$, $\alpha^* = -2\mu$, where μ is the rigidity modulus. Comparing equations (4.2) and (4.8) we have $\eta^* = 0$. Equations (4.5) and (4.20) give $\eta = -2v\tau$. Then equation (4.7) becomes, with $dq/dt = 0$,

$$a^2 \frac{d\rho}{dt} = \frac{dp_x}{dt} - 2\mu \left(1 - \frac{\Gamma^*\tau}{\mu}\right) \frac{d\epsilon_x^p}{dt}.$$

(4.21)

Since τ is equal to half the yield stress, the effect of including the thermal variables is to slightly decrease the effective relaxation function[20].

(ii) *Phase transitions in liquids.* These have been treated in detail elsewhere[21, 22]. In the present formalism, if λ is the fraction of material in phase 2, the parameters of equation (4.1) are [23]:

$$a^{*2} = \left.\frac{\partial P}{\partial \rho}\right|_{S,\lambda} = v^2/(\beta v - T\gamma^2 v^2/C_{P\lambda}),$$

(4.22)

where

$$\gamma = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_{P,\lambda},$$

$$C_{P\lambda} = (1-\lambda)C_{P1} + \lambda C_{P2},$$

$$C_{Pi} = \text{sp. heat at constant pressure of phase } i,$$

$$\beta = -\frac{1}{v}\left(\frac{\partial v}{\partial P}\right)_{T,\lambda},$$

$$\Gamma^* = \gamma v^2/(\beta v C_{P\lambda} - \gamma^2 v^2 T),$$

(4.23)

$$\alpha^* = \left.\frac{\partial P}{\partial \lambda}\right|_{v,S} = \frac{\gamma v T \Delta S - C_{p\lambda}\Delta v}{\gamma^2 v^2 T - \beta v C_{p\lambda}}$$

(4.24)

Equation (4.2) is unchanged: $\dot{w} = -P^*\dot{v}$, i.e. $\eta^* = 0$. Equation (4.3) becomes

$$\dot{e} = -P\dot{v} + T\dot{S} + (\mu_2 - \mu_1)\dot{\lambda}$$

where μ_1 and μ_2 are the chemical potentials of phases 1 and 2. Then

$$\eta = \mu_2 - \mu_1. \tag{4.25}$$

In equation (4.7), $p_x^* = p_x = P$, $\eta^* = 0$, a^*, Γ^*, α^* and η are given by equations (4.22)–(4.25).

5. DISCUSSION

Equations (3.5) and (4.7) show the effects of introducing a variable other than v and S in the constitutive equation of material through which a plane wave is passing. The consequence is always to add an additional term to the $\dot{p}$, $\dot{\rho}$, $\dot{S}$ relation. This additional term is carried through the flow equations and invariably introduces 'Maxwell attenuation' into the shock decay process. The existence of this term does not depend on the additional process being dissipative, though its coefficient is enhanced by dissipation, which causes flow behind the shock to be entropic.

The fundamental reason for this effect lies in the fact that there are but three equations to describe continuum flow and four variables, p, ρ, S, u. A single constitutive relation then provides a soluble set, as long as no new variable is introduced. If it is, it must be carried through the equations in an *ad hoc* way. Only if another constitutive relation and a corresponding differential equation are added, can this situation be avoided. For example, if electric displacement is added, as in equation (3.20), the Maxwell-like term can be eliminated if an additional constitutive relation and the appropriate electromagnetic equations are added.

When the extra variable is a density derivative, as for the viscous fluid, a curious situation arises. One can proceed, in principle, to combine the constitutive relation, equation (3.21), with the flow equations, eliminate p and u, and solve the resulting third order equation for ρ. But in so doing the utility of heuristic understanding of wave propagation problems is lost and the new problem stands in isolation. Yet one feels that the problem as stated is still essentially a wave propagation problem, and that only a little tinkering with the mathematical apparatus of wave propagation should provide understanding. This view may be wrong; certainly it has not yet yielded an estimate of the importance of viscous attenuation in the shock decay process or a satisfactory description of its interaction with hydrodynamic attenuation.

The reader should be aware that an extensive literature exists of which the principal purpose is to determine the entire propagation history of decaying shock waves, principally in gases. The problem has little relation to the one discussed here, but the interested reader can gain entry to the literature through Ref. [9, p. 160].

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